Solution: Problem Set 1
Calculus 1

October 16, 2011

Problem 1

Before we sketch the function, let's exploit some of the obvious properties of f. First notice that since $f(x)$ is the square of the number $1/x$, therefore $f(x)$ is always non-negative. This means that the graph of f is never below the y-axis. We also see that when x is very large $1/x^2$ is very small and when x is close to zero, $1/x^2$ is a very large number. In fact $f(x)$ becomes unbounded as x gets close to 0. Also, $f(-x) = f(x)$, i.e. f is an even function of x which means that its graph is symmetric about the y-axis.

In view of all these properties, now its trivial to sketch f and its plot is seen in the Figure 1.

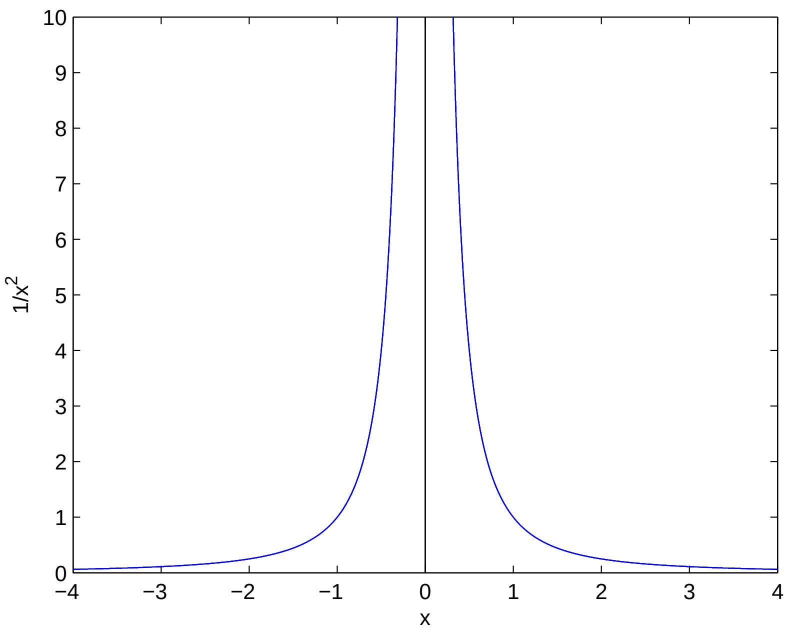

Figure 1: $f(x) = 1/x^2$.

(a) We claim that $\lim_{x\to\infty} f(x) = 0$. This is justified by observing that if x is chosen large enough then $f(x)$ can be made as close to zero as we desire.

(b) $\lim_{x\to 0} 1/x^2 = \infty$. Remember that ∞ is not a real number. This limit must be understood in the sense that if x is made close enough to zero, $1/x^2$ can me made larger than any positive number. Also notice that even if we extend the real numbers by including the symbol ∞ to represent a process in which a number can be made larger than any given number, $\lim_{x\to 0} 1/x$ still does not exist because $1/x$ attains arbitrarily large positive as well as negative numbers in every neighbourhood of $x = 0$.

(c) $f(0)$ is undefined.

Problem 2

For $x < 4$ the function is defined by a first degree polynomial and hence represents a straight line, which has a negative slope in this case. For $x > 4$ the function is simply a 'shifted' version of $\sqrt{x}$. We can clearly see that $\lim_{x\to 4} f(x) = 0$ and the argument to support this claim is simple: we can make the function values as close to 0 as we want by selecting a small enough neighbourhood around the point $x = 4$.

Although $\lim_{x\to 4} f(x)$ exists, the value of the function at $x = 4$, i.e. $f(4)$ is NOT defined.

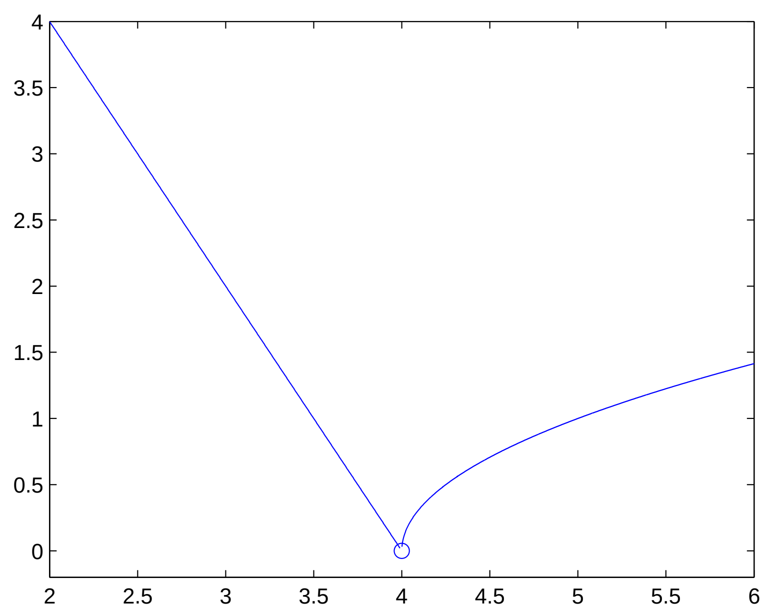

Figure 2: A sketch of the function defined in Problem 2.

Problem 3

We are interested in finding the limit

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$$

and we can see that substituting $t = 0$ gives us an indeterminate $0/0$ form. We therefore try to see if the expression above can be written an alternative form. If we multiply the numerator and the denominator by

$\sqrt{t^2+9}+3$, we get

$$\frac{\sqrt{t^2+9}-3}{t^2} = \frac{t^2}{t^2(\sqrt{t^2+9}+3)}$$

Now we cancel t^2 from the expression, keeping in mind that this means t^2 and hence t can not be 0. We are then left with the problem of finding the limit

$$\lim_{t\to 0} \frac{1}{(\sqrt{t^2+9}+3)}$$

Now, we can see that as t is getting closer and closer to 0 (but not equal to 0), the expression above is getting closer and closer to the number 1/6 and we can make the above expression as close to 1/6 as we desire by making t close enough to 0, therefore, we say

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0} \frac{1}{(\sqrt{t^2+9}+3)} = 1/6.$$

Problem 4

To sketch $f(x) = 1/(x^2+1)$ notice that the function is even, always positive and gets closer and closer to zero as x becomes large in magnitude on either side of the real line. The derivative of f is given as

$$f'(x) = \frac{d}{dx}[f(x)] = \frac{-2x}{(x^2+1)^2}$$

We can see that at $x = 0$ the derivative *vanishes* i.e. the derivative becomes 0 at $x = 0$. Recall that the derivative of a function at a point gives us the slope of the tangent at that point. Therefore, the function has a horizontal tangent at $x = 0$. Keeping in view all of these observations, we can now see that f should look like as shown in Figure 3.

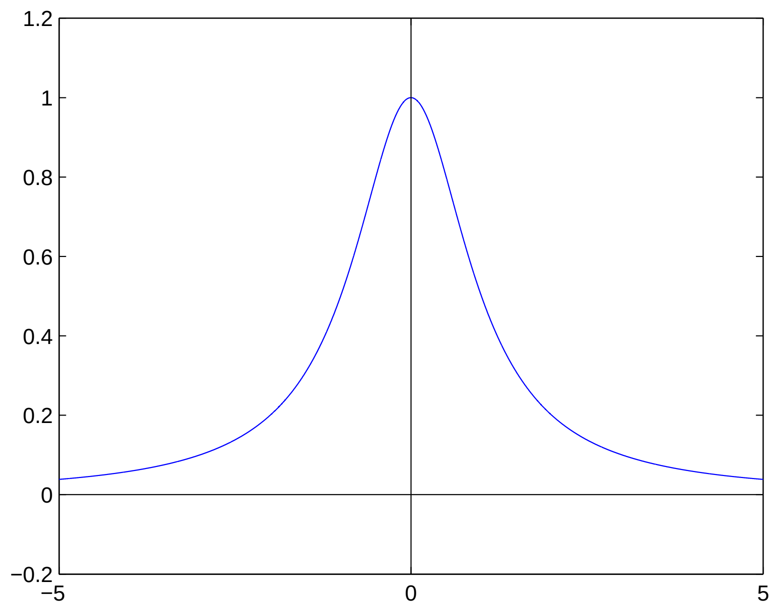

Figure 3: $f(x) = 1/(x^2+1)$.

To find the equation of the tangent line at $x = 2$, we calculate the derivative of f at $x = 2$. We get

$$f'(2) = -\frac{4}{25}$$

Now we can easily find the equation for the tangent since we now know its slope. Appealing to the point-slope form of a line, we know that a line passing through the point (x_0, y_0) with a slope m is given by

$$y - y_0 = m(x - x_0).$$

In our case, $x_0 = 2, y_0 = f(2) = 1/5$ and $m = -4/25$. Using these values we get the equation for the tangent line as

$$y = -\frac{4}{25}(x - 2) + \frac{1}{5}.$$

We can see this tangent line in Figure 4.

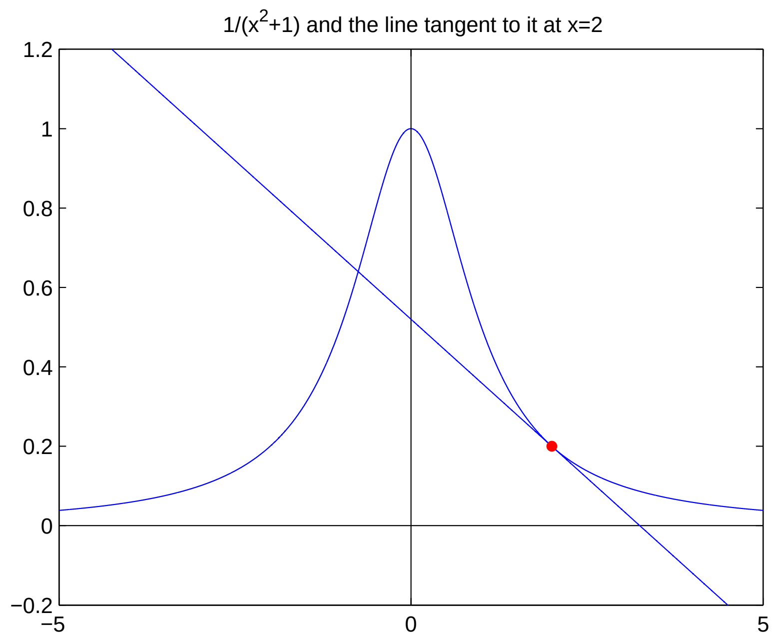

Figure 4: $f(x) = 1/(x^2 + 1)$ and the line tangent to it at $x = 2$.

Now we try to sketch the derivative of f. We already know that $f'(x) = \frac{-2x}{(x^2+1)^2}$. Again, notice that this is an odd function of x since $f(-x) = -f(x)$. This tells us that the sketch should be symmetric about the origin. Also, $f'(x)$ is negative when $x > 0$ and positive when $x < 0$. The derivative is 0 when $x = 0$. From the expression of f' we can also see that the derivative becomes closer and closer to zero as x becomes very large in magnitude on either side of the real line. With the help of all these observations, we now sketch f and its derivative together in Figure 5.

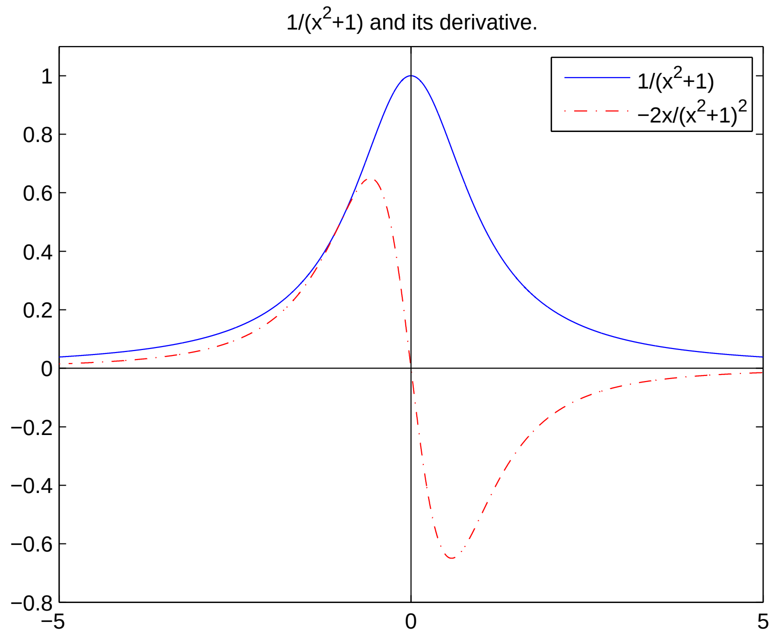

Figure 5: $f(x) = 1/(x^2+1)$ and its derivative $f'(x) = \frac{-2x}{(x^2+1)^2}$.

Problem 5

The sketch of the function is shown in Figure 6. We can easily see that $\lim_{x\to a} f(x)$ exists for every a except when $a = \pm 1$.

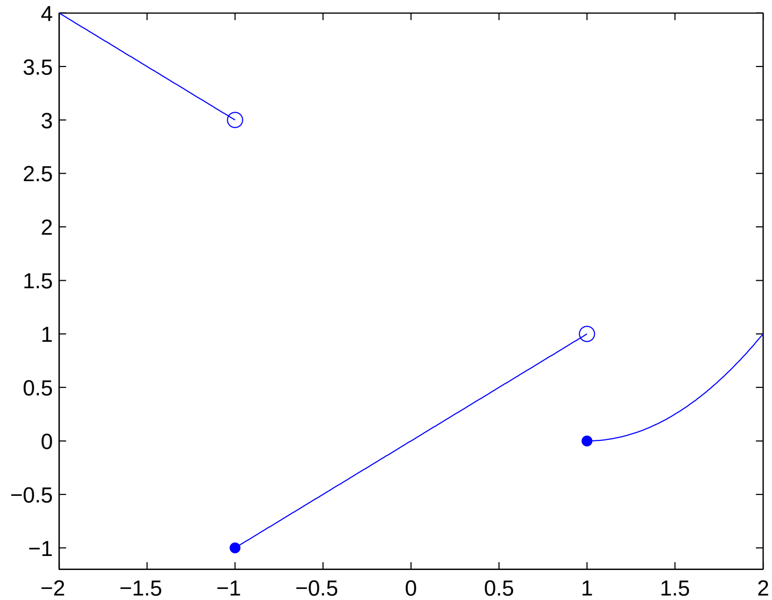

Figure 6: Piecewise function defined in Problem 5.

Problem 6

First of all note that we have to *disprove* the statement. The statement is: if $f(x) < g(x)$ for all x, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$. Recall, that this statement is of the form A implies B, i.e.

$$A \implies B.$$

To disprove it, we need to show an example where A is true but B is false, which means for our case that we need to find functions f and g such that $f(x) < g(x)$ for all x but $\lim_{x\to a} f(x) \geq \lim_{x\to a} g(x)$.

There can be many examples to achieve this task, but let us define the function f as

$$f(x) = 0, \text{ for all } x.$$

and g as

$$g(x) = \begin{cases} |x|, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

A plot of the function is shown in Figure 7. We can see that f is simply the horizontal line along the x-axis, while g is $|x|$ except at $x = 0$, where we have deliberately defined it to take the value 1. This ensures that $f(x) < g(x)$ for all x.

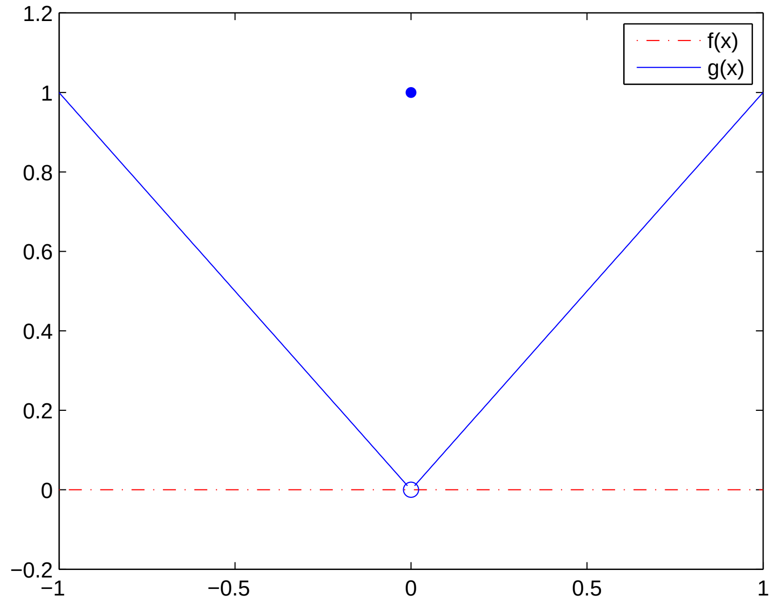

Figure 7: $f(x) < g(x)$ for all x but $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x)$.

However, we can also see that $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$. This disproves the original statement since both limits are equal.

Problem 7

Again, there is no unique solution to this problem and one can come up with a lot of different examples. Consider the function f defined as

$$f(x) = \frac{1}{x+1}.$$

Then $f(x^2) = 1/(x^2+1)$. We can see from the definition of f that $\lim_{x\to -1} f(x)$ does not exist but $\lim_{x\to -1} f(x^2)$ exists and equals $1/2$.