Solution: Problem Set 2
Calculus 1

October 26, 2011

Problem 1

The sketch of the function is trivial and shown in Figure 1. $\lim_{x \to 0} f(x)$ does not exist and hence the function is not continuous at $x = 0$.

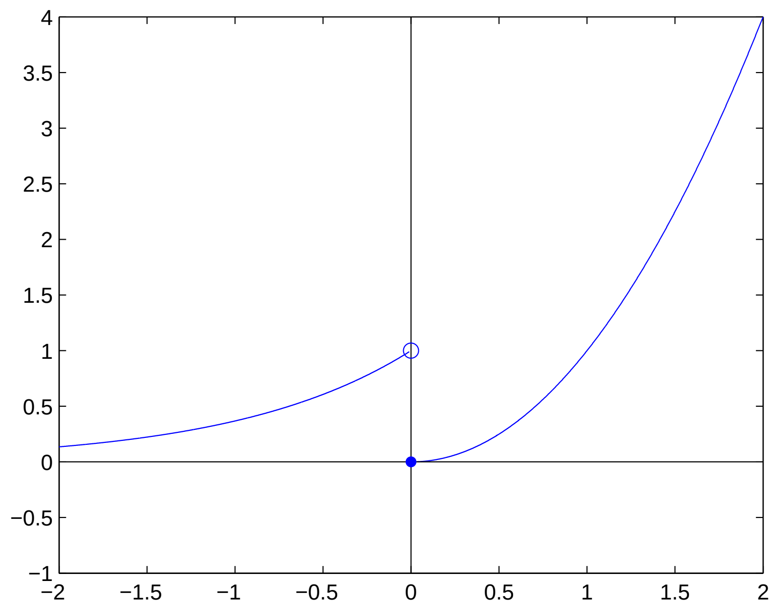

Figure 1: f of Problem 1.

Problem 2

For $x \neq 1$, we have

$$f(x) = \frac{x(x-1)}{(x+1)(x-1)} = \frac{x}{x+1} = \frac{(x+1)-1}{x+1} = 1 - \frac{1}{x+1}.$$

The graph of $1-\frac{1}{x+1}$ is similar to the graph of $\frac{1}{x}$ with three important differences. (1). Instead of becoming unbounded at $x=0$, the function f becomes unbounded at $x=-1$. (2). The function tends to 1 as x becomes large in magnitude. (3). The two branches of the graph of f are inverted due to the minus sign in $1-\frac{1}{x+1}$. The graph can be seen in Figure 2.

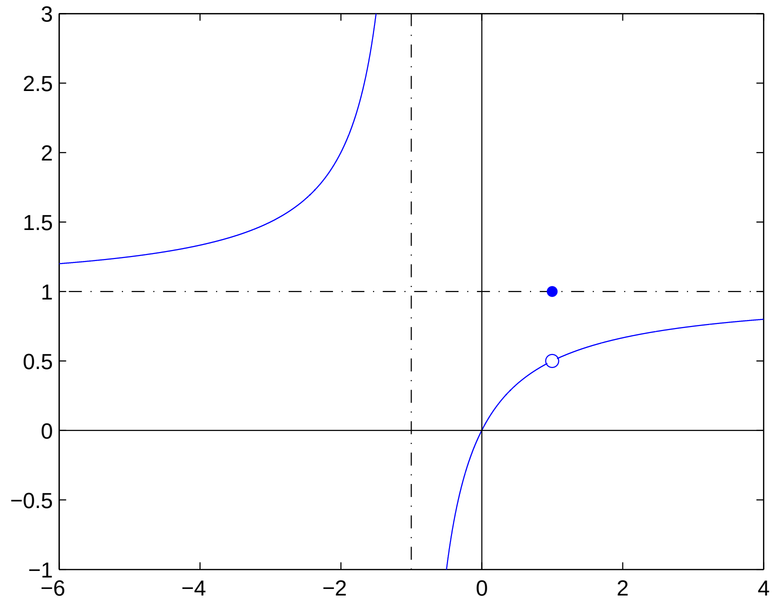

Figure 2: f of Problem 2.

At $x=1$ the function is defined to take the value 1, however, we can clearly see that

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(1-\frac{1}{x+1}\right) = \frac{1}{2}.$$

Since $\lim_{x\to 1} f(x) \neq f(1)$, f is not continuous at $x=1$.

Problem 3

Consider the function f defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

Then we see that f is discontinuous for every x because at each x the limit of the function does not exist. However, $|f|$ is clearly continuous for every x since $|f(x)|$ is identically equal to 1.

Problem 4

Consider the function f defined as

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n, n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

This function is discontinuous at each x such that $x = 1/n$, where n is an integer because $\lim_{x \to 1/n} f(x) = 0 \neq 1 = f(1/n)$. However for all other x, the function is continuous.

Problem 5

If $f(x) = \ln(x + \sqrt{(x^2+1)}$, then

$$\begin{aligned} f(-x) &= \ln(-x + \sqrt{((-x)^2 + 1)}) \\ &= \ln\left((-x + \sqrt{(x^2+1)}) \left[\frac{x + \sqrt{(x^2+1)}}{x + \sqrt{(x^2+1)}} \right] \right) \\ &= \ln\left(\frac{1}{x + \sqrt{(x^2+1)}} \right) \\ &= -\ln(x + \sqrt{(x^2+1)}) \qquad (\text{since } \ln(1/x) = -\ln(x)) \\ &= -f(x). \end{aligned}$$

Using the chain rule repeatedly, we get

$$\begin{aligned} f'(x) &= \frac{1}{x + \sqrt{(x^2+1)}} \left(1 + \frac{2x}{2\sqrt{(x^2+1)}} \right) \\ &= \frac{1}{x + \sqrt{(x^2+1)}} \left(\frac{\sqrt{(x^2+1)} + x}{\sqrt{(x^2+1)}} \right) \\ f'(x) &= \frac{1}{\sqrt{(x^2+1)}} > 0. \end{aligned}$$

Since $f'(x) > 0$ for all x, f is an increasing function on the whole real line.

Sketching the function is now fairly straight forward. Since f is an odd function, it must pass through the origin. We can confirm this by noticing that $f(0) = 0$. f is an increasing function but its tangents become almost horizontal as x increases in magnitude. This gives us a sketch similar to Figure 3.

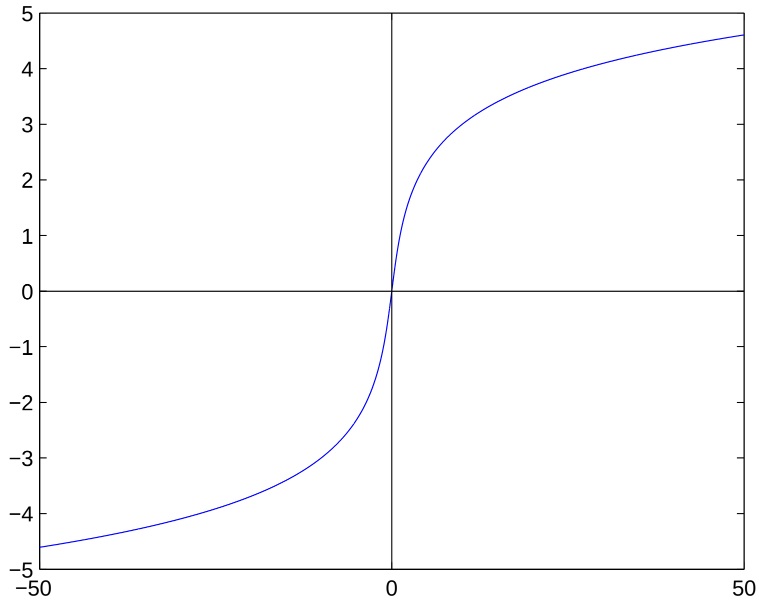

Figure 3: $f(x) = \ln(x + \sqrt{(x^2+1)}$

To calculate the inverse function of f, let $y = f(x) = \ln(x + \sqrt{(x^2+1)}$. Then

$$\begin{aligned} e^y &= x + \sqrt{x^2+1} \\ (e^y - x)^2 &= x^2 + 1 \\ e^{2y} - 2xe^y - 1 &= 0 \\ x &= \frac{e^y - e^{-y}}{2}. \end{aligned}$$

The inverse function (as a function of the variable x) can be written as

$$f^{-1}(x) = \frac{e^x - e^{-x}}{2} = \sinh(x).$$

The sketch of f^{-1} is simply obtained by reflecting f about the line $y = x$, or we can see that for large positive x, f^{-1} is similar to $e^x/2$ and for negative numbers with large magnitude, f^{-1} is similar to $e^{-x}/2$. The graph of f^{-1} can be seen in Figure 4

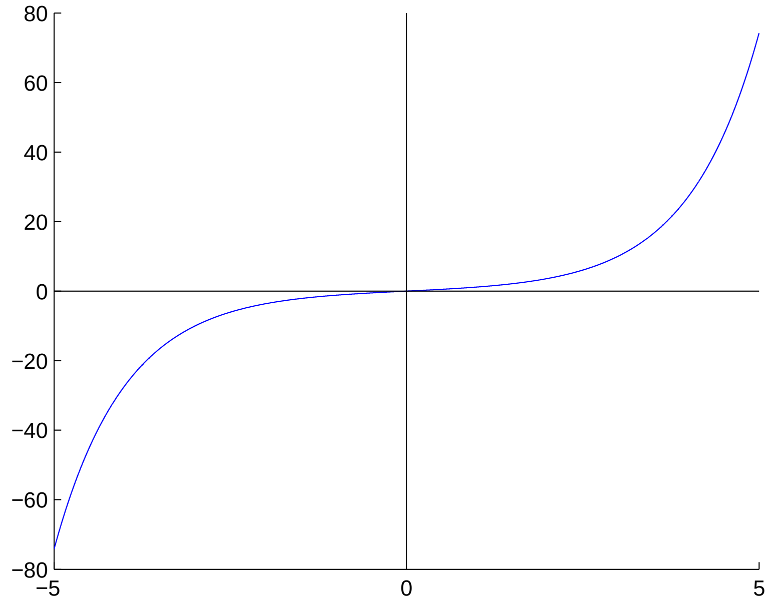

Figure 4: $f(x) = \sinh(x)$

Problem 6

Let us take time on the x-axis and position on the y-axis. We designate M as the position of the monastery and T as the position of the mountain top. Then the journey of the monk from monastery to the mountain top can be represented as a function of time. We call this function u (u for going "up"). Similarly, once at the top, the monk's way back from T to M can be represented as a function say d (d for going "down"). Both these functions u and d can be seen in Figure 5.

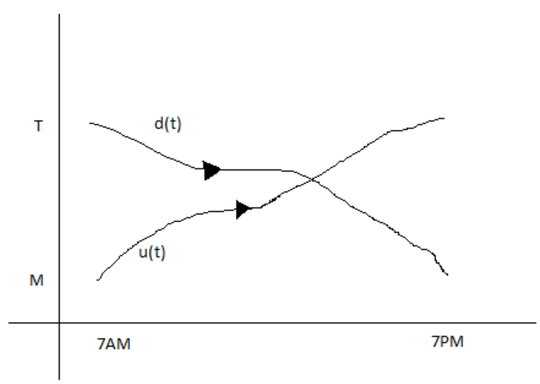

Figure 5: The monk going up and down.

Now consider the function $u-d$. At 7AM, $u(7AM)-d(7AM)<0$; while at 7PM, $u(7PM)-d(7PM)>0$. Furthermore, $u-d$ is continuous since u and d both are continuous. Then, by the intermediate value property, there exists a time t between 7AM and 7PM, such that $u(t)-d(t)=0$ or in other words the monk crosses a point of his journey at the same time instance.

Problem 7

Let $\overline{PQ}=h=\overline{ST}$ and let $\overline{QT}=D$. Suppose further that $\overline{QR}=x$, then $\overline{RT}=D-x$. The total length L of the rope is given by

$$L(x)=\sqrt{h^2+x^2}+\sqrt{h^2+(D-x)^2}.$$

Then to find a minimum of L, we first differentiate L to get

$$\frac{dL}{dx}=\frac{x}{\sqrt{h^2+x^2}}+\frac{-(D-x)}{\sqrt{h^2+(D-x)^2}}$$

We now put $\frac{dL}{dx}=0$. This implies

$$\begin{aligned}
\frac{x}{\sqrt{h^2+x^2}}+\frac{-(D-x)}{\sqrt{h^2+(D-x)^2}} &= 0\\
\frac{(D-x)}{\sqrt{h^2+(D-x)^2}} &= \frac{x}{\sqrt{h^2+x^2}}\\
(D-x)^2(h^2+x^2) &= x^2(h^2+(D-x)^2)\\
x^2(D-x)^2+h^2(D-x)^2 &= x^2h^2+x^2(D-x)^2\\
(D-x)^2 &= x^2 \qquad \text{since } h\neq 0\\
(D-x)^2-x^2 &= 0\\
(D-x+x)(D-x-x) &= 0\\
D(D-2x) &= 0\\
x &= D/2 \qquad \text{since } D\neq 0.
\end{aligned}$$

We should now check that $L''(D/2)<0$ to make sure that L is minimized at $x=D/2$. This is left as an exercise. The solution $x=D/2$ has a simple geometric meaning, namely, triangle PQR is congruent to triangle STR and hence angles a and b are also equal.

Problem 8

(i) $Df(2x+1)=f'(2x+1)=\cos(2x+1)$.

(ii) $D(f\circ h)(x)=Df(h(x))Dh(x)=\cos(h(x))(2)=2\cos(2x+1)$.

Problem 9

By the mean value theorem, we have

$$\frac{f(1001)-f(1)}{1001-1}=f'(c), \tag{1}$$

where c is a number in the interval $(1, 1001)$. Since $f(1) = 0$, (1) can be rewritten as

$$f(1001) = 1000 f'(c). \tag{2}$$

Now we don't know what the value of c is but we know that $f'(c)$ is greater than or equal to the minimum value of $f'(x)$, where x is in $(1, 1001)$. We therefore have

$$f'(c) \geq \min f'(x), \quad x \in (1, 1001). \tag{3}$$

We now search for the minimum value of $f'(x)$. Since $f'(x) = \frac{1}{\sin\left(\frac{\pi}{5+x^2}\right)}$, $f'(x)$ will be minimum when $\sin\left(\frac{\pi}{5+x^2}\right)$ is maximized. We notice that sin( ) is an increasing function on $[0, \pi/2]$, so the maximum value of sin will be attained when its argument is maximum. As x varies in the interval $(1, 1000)$, $\frac{\pi}{5+x^2}$ varies from $\pi/6$ to $\pi/1000005$ and clearly the maximum value is $\pi/6$. So the maximum of $\sin\left(\frac{\pi}{5+x^2}\right)$ is attained when $x = 1$ and

$$\sin\left(\frac{\pi}{5+x^2}\right)\Bigg|_{x=1} = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}. \tag{4}$$

Hence the minimum value of $f'(x) = 2$. This allows us to rewrite (3) as

$$f'(c) \geq 2 \tag{5}$$

Multiplying both sides of the above inequality by 1000 and then comparing with (2), we get the required result as

$$f(1001) \geq 2000. \tag{6}$$