Solution: Problem Set 3
Calculus 1

November 13, 2011

Problem 1

The distance l between the point (x_0, y_0) and a point on the line $f(x) = mx + b$ is given by the *distance* formula:

$$L = l^2 = (x - x_0)^2 + (mx + b - y_0)^2 \tag{1}$$

Taking the derivative of L with respect to x and equating it with 0, we get

$$L'(x) = 2(x - x_0) + 2m(mx + b - y_0) = 0.$$

Solving for x gives

$$x = \frac{m(y_0 - b) + x_0}{m^2 + 1}. \tag{2}$$

Before using this value to find the minima, we calculate the second derivative of L:

$$L''(x) = 2 + 2m^2.$$

We can clearly see that $L''(x) > 0$ for all values of m, regardless of x. So, we can be sure that we are finding the minima and not the maxima.

All we have to do now is to put the value of x given by (2) in (1) and simplify. This is left as an exercise!

Problem 2

Let us assume that $x \in [-1, 1]$, otherwise $\sin^{-1}(x)$ is not a real number. Since $g(x) = \sin^{-1}(x)$, we can write

$$\sin(g(x)) = x.$$

We assume here that $g(x) \in [-\pi/2, \pi/2]$. Let $f(x) = \sin(x)$. Then the last equation can be written as

$$(f \circ g)(x) = x.$$

Differentiating both sides with respect to x, we get

$$\begin{aligned}
D(f \circ g)(x) &= 1 \\
Df(g(x))Dg(x) &= 1 \\
\cos(g(x))Dg(x) &= 1 \\
Dg(x) &= \frac{1}{\cos(g(x))} \\
Dg(x) &= \frac{1}{\sqrt{1 - [\sin(g(x))]^2}} \\
Dg(x) &= \frac{1}{\sqrt{1 - x^2}}.
\end{aligned}$$

Problem 3

Since $f'(x) = 2 - \sin(x) > 0$ for all x, $f(x)$ is an increasing function and hence one-to-one. We can see that the graph of an increasing or a decreasing function intersects a horizontal line at most once and hence represents a one-to-one function.

Since $f(0) = 1$, $f^{-1}(1) = 0$.

Problem 4

The graph of $x + \lfloor x \rfloor$ is shown in Figure 1

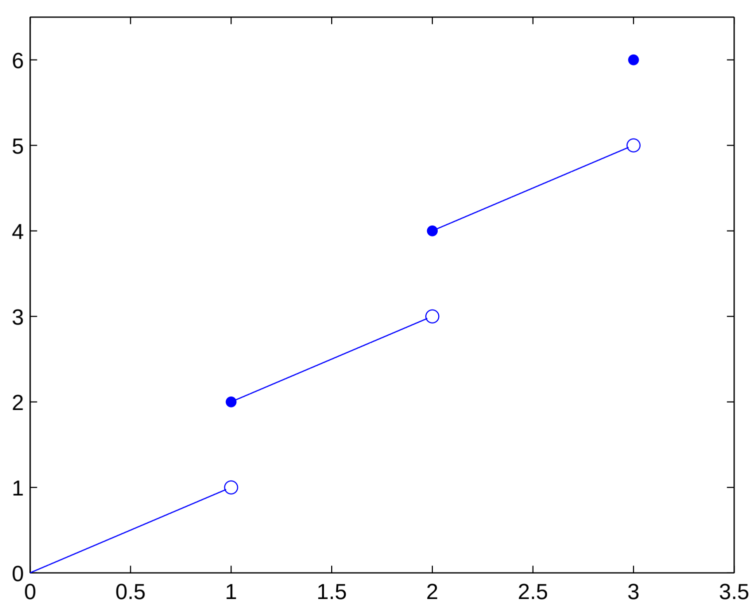

Figure 1: $x + \lfloor x \rfloor$.

First we need to see whether this function is integrable or not. We can clearly see that the this function is continuous except at $x = 1, x = 2$ and $x = 3$ where it has finite jump discontinuities. From our lecture notes, we know that a function which is otherwise continuous but has a finite jump discontinuity at a point is integrable; this is true because the upper and lower sum can be made as close to each other as we please. Therefore, the function in question is integrable.

To compute the integral, recall that all we need to do is to find the area under the curve. This can be done geometrically in this simple case by dividing the area under the curve as shown in Figure 2.

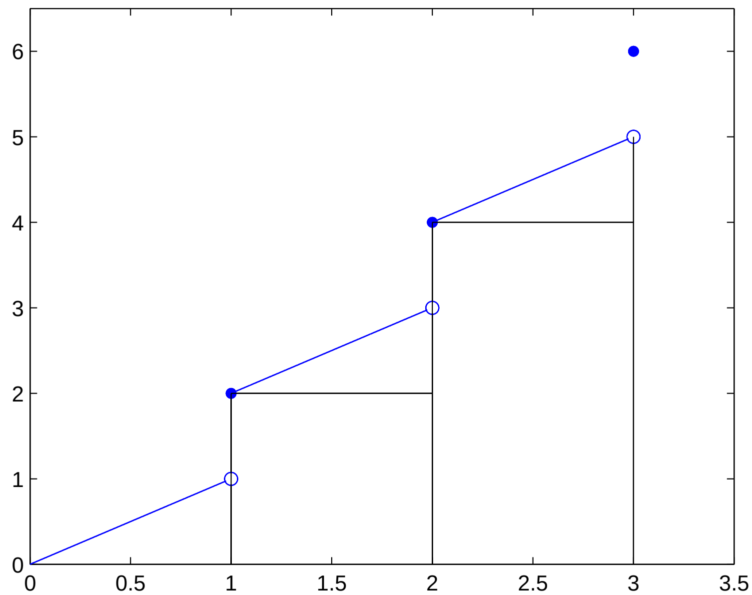

Figure 2: Area under the curve for $x + \lfloor x \rfloor$.

We can see that there are three triangles each with area 1/2 and two rectangles with areas 2 and 4. The total are under the curve is thus $3(1/2) + 2 + 4$ and hence

$$\int_0^3 f = 7\frac{1}{2}.$$

Problem 5

The graph of f and g can be seen in Figure 3. Since $f(x) > g(x)$ for the domain in question, the area between the graphs can be easily calculated by integrating the difference of f and g.

Also if $x < 1$, then $x - 1 < 0$ and $|x - 1| = -(x - 1) = 1 - x$. For $x > 1$, $x - 1 > 0$ and $|x - 1| = x - 1$.

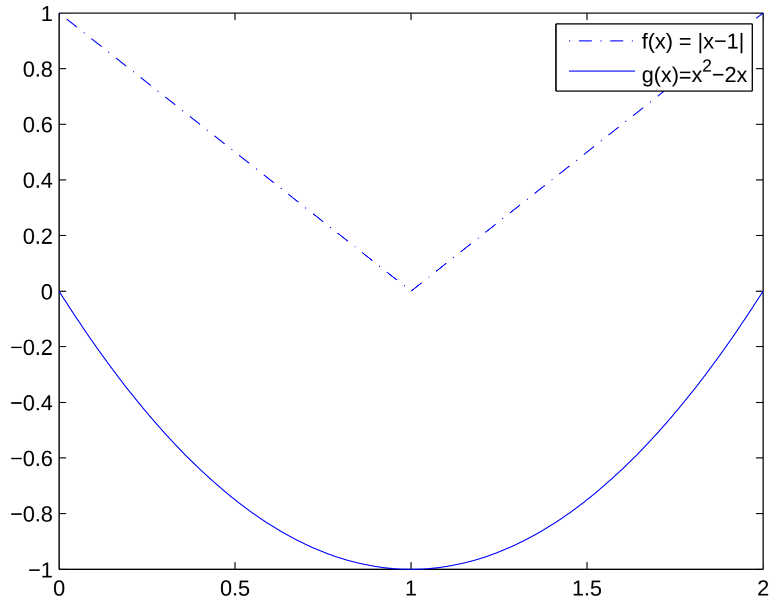

Figure 3: Area between the graphs of f and g.

$$\begin{aligned} Area = \int_0^2 f - g &= \int_0^2 \left(|x-1| - (x^2 - 2x)\right) dx \\ &= \int_0^1 \left(1 - x - x^2 + 2x\right) dx + \int_1^2 \left(x - 1 - x^2 + 2x\right) dx \\ &= \frac{7}{6} + \frac{7}{6} \\ &= \frac{7}{3} \end{aligned}$$

Problem 6

(a)

Let $x = 3\sin(u)$, then $dx = 3\cos(u)du$ and $\sqrt{9-x^2} = 3\cos(u)$. Also when $x = -3, u = -\pi/2$ and when $x = 3, u = \pi/2$. Using all these we get

$$\begin{aligned}\int_{-3}^{3}\sqrt{9-x^2} &= \int_{-\pi/2}^{\pi/2}(3\cos(u))(3\cos(u))du \\ &= 9\int_{-\pi/2}^{\pi/2}\cos^2(u)du \\ &= 9\int_{-\pi/2}^{\pi/2}\frac{1+\cos(2u)}{2}du \\ &= \frac{9\pi}{2}\end{aligned}$$

(b)

Let $x = 2\sin(u)$, then $dx = 2\cos(u)du$ and $\sqrt{1-\frac{1}{4}x^2} = \cos(u)$. Also when $x = 0, u = 0$ and when $x = 2, u = \pi/2$. Substituting all these we have

$$\begin{aligned}\int_{0}^{2}\sqrt{1-\frac{1}{4}x^2} &= \int_{0}^{\pi/2}(\cos(u))(2\cos(u))du \\ &= 2\int_{0}^{\pi/2}\cos^2(u)du \\ &= 2\int_{0}^{\pi/2}\frac{1+\cos(2u)}{2}du \\ &= \frac{\pi}{2}\end{aligned}$$

(c)

Let $x = 2\sin(u)$, then $dx = 2\cos(u)du$, when $x = -2, u = -\pi/2$ and when $x = 2, u = \pi/2$. Also $(x-3)\sqrt{4-x^2} = (2\sin(u)-3)(2\cos(u))$. Substituting all these we get

$$\begin{aligned}\int_{-2}^{2}(x-3)\sqrt{4-x^2}dx &= \int_{-\pi/2}^{\pi/2}(2\sin(u)-3)(2\cos(u))2\cos(u)du \\ &= 8\int_{-\pi/2}^{\pi/2}\cos^2(u)\sin(u)du - 12\int_{-\pi/2}^{\pi/2}\cos^2(u)du \\ &= -8\frac{\cos^3(u)}{3}\Big|_{-\pi/2}^{\pi/2} - 12\int_{-\pi/2}^{\pi/2}\frac{1+\cos(2u)}{2}du \\ &= -6\pi\end{aligned}$$

Problem 7

(a), (b) and (d) all have a non integrable singularity in the interval of integration and therefore the integral does not exist. For part (c), recall form lecture notes that

$$\begin{aligned}\int_1^\infty f &= \lim_{N\to\infty}\int_1^N f\\ &= \lim_{N\to\infty}\int_1^N \frac{1}{x^2}dx\\ &= \lim_{N\to\infty}\frac{-1}{x}\Big|_1^N\\ &= \lim_{N\to\infty}\left(1-\frac{1}{N}\right)\\ &= 1\end{aligned}$$

Problem 8

By the corollary to the fundamental theorem,

$$\int_1^4 Df = f(4) - f(1).$$

Since $\int_1^4 Df = 14$ and $f(1) = 12$, We therefore get

$$14 = f(4) - 12$$

and hence $f(4) = 26$.

Problem 9

$$0 \le \frac{x^2}{x^4+x^2+1} \le \frac{x^2}{x^4} = \frac{1}{x^2}.$$

Integrating from 5 to 10, we get

$$0 \le \int_5^{10}\frac{x^2}{x^4+x^2+1} \le \int_5^{10}\frac{1}{x^2} = 0.1.$$

Problem 10

FSc. Style: Let $2y+1 = u$, then $dy = du/2$. Also, when $y = 0, u = 1$ and when $y = 4, u = 9$. Using these values we get

$$\int_0^4 \sqrt{2y+1}dy = \frac{1}{2}\int_1^9 \sqrt{u}du = \frac{u^{\frac{3}{2}}}{3}\Big|_1^9 = \frac{26}{3}.$$

New Style: Let $g(x) = \sqrt{x}$ and $h(x) = 2x+1$ so that $(g\circ h)(x) = g(h(x)) = \sqrt{2x+1}$. Also $h'(x) = 2$. The theorem in the lecture notes says

$$\int_a^b (g\circ h)(x)h'(x)dx = \int_{h(a)}^{h(b)} g(x)dx.$$

Therefore, if we put $a = 0$ and $b = 4$ in the above equation, we get

$$\int_0^4 \sqrt{(2x+1)}(2)dx = \int_{h(0)}^{h(4)} \sqrt{x}dx.$$

This implies

$$\int_0^4 \sqrt{2x+1}dx = \frac{1}{2}\int_1^9 \sqrt{x}dx = \frac{26}{3}.$$

Problem 11

We have to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any choice of partition $P : x_0 < x_1 < ... < x_{n-1} < x_n$ where $x_0 = 0$ and $x_n = b$ and $\|P\| < \delta$,

$$|\sum_{k=1}^{n} f(c_k)\Delta x_k - \frac{b^2}{2}| < \epsilon \tag{3}$$

with each $\Delta x_k < \delta$. Note that $\Delta x_k = x_k - x_{k-1}$ and $c_k \in [x_{k-1}, x_k]$. Since the function is increasing, it is obvious that

$$L(f, P) \le \sum_{k=1}^{n} f(c_k)\Delta x_k \le U(f, P)$$

So in order to prove (3) it is sufficient to prove that

$$\left|U(f, P) - \frac{b^2}{2}\right| < \epsilon \text{ and } \left|\frac{b^2}{2} - L(f, P)\right| < \epsilon$$

Why? Because if these inequalities are true it means that $U(f, P)$ and $L(f, P)$ are in an ϵ-neighbourhood of $b^2/2$. Since the Riemann sum is between these two numbers, it means that the Riemann sum is also in an ϵ-neighbourhood of $b^2/2$.

Now since the function is strictly increasing, upper and lower sums are obtained by evaluating the function at the upper and lower endpoints of the subintervals, respectively.

Considering the upper sum, hence, the inequality comes to

$$|\sum_{k=1}^{n} f(x_k)\Delta x_k - \frac{b^2}{2}| = |\sum_{k=1}^{n} x_k\Delta x_k - \frac{b^2}{2}| < \epsilon$$

We can write $b^2/2$ as

$$\frac{1}{2}\sum_{k=1}^{n}(x_{k-1} + x_k)\Delta x_k$$

because

$$\begin{aligned}
\sum_{k=1}^{n}(x_{k-1} + x_k)\Delta x_k &= \sum_{k=1}^{n}(x_{k-1} + x_k)(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n}(x_k^2 - x_{k-1}^2) \\
&= x_n^2 - x_0^2 \\
&= b^2
\end{aligned}$$

So we have to prove

$$\Big|\sum_{k=1}^{n} x_k\Delta x_k - \frac{1}{2}\sum_{k=1}^{n}(x_{k-1}+x_k)\Delta x_k\Big| < \epsilon$$

Now

$$\begin{aligned}\Big|\sum_{k=1}^{n} x_k\Delta x_k - \frac{1}{2}\sum_{k=1}^{n}(x_{k-1}+x_k)\Delta x_k\Big| &= \Big|\sum_{k=1}^{n} \Delta x_k\big(x_k - \frac{1}{2}(x_{k-1}+x_k)\big)\Big| \\ &= \Big|\sum_{k=1}^{n} \Delta x_k\frac{(x_k - x_{k-1})}{2}\Big|\end{aligned}$$

This can be easily simplified to

$$\begin{aligned}\Big|\frac{1}{2}\sum_{k=1}^{n}\Delta x_k(x_k - x_{k-1})\Big| &\leq \Big|\frac{1}{2}\sum_{k=1}^{n}\delta(x_k - x_{k-1})\Big| \\ &= \frac{1}{2}\delta\Big|\sum_{k=1}^{n}(x_k - x_{k-1})\Big| \\ &= \frac{1}{2}\delta(x_n - x_0) \\ &= \frac{1}{2}b\delta\end{aligned}$$

The first inequality is true because $\Delta x_k \leq \delta$ for all x_k. Now

$$\frac{b\delta}{2} < \epsilon \text{ if } \delta < \frac{2\epsilon}{b}$$

. So for upper sum, we get

$$|U(f,P) - \frac{b^2}{2}| < \epsilon \text{ if } \delta < \frac{2\epsilon}{b}$$

The same procedure can be followed for the lower sum to arrive at the same result as above i.e.

$$|L(f,P) - \frac{b^2}{2}| < \epsilon \text{ if } \delta < \frac{2\epsilon}{b}$$

Hence these two equations imply that

$$\Big|\sum_{k=1}^{n} f(c_k)\Delta x_k - \frac{b^2}{2}\Big| < \epsilon$$

when $\delta < 2\epsilon/b$.