University of Engineering and Technology Lahore
Department of Electrical Engineering

MA-100 **Problem Set #3** October 24, 2011

Due: Tuesday, November 1.

Reading

- Calculus notes at Star Photocopier (this week's notes will be in two stages).
- Relevant portions of the F.Sc/A-level's book

Announcements

- Office hours for the rest of the semester:
  Monday, Wednesday, Thursday: 12-1 and 3-4.
- In this course, we assume that you have seen calculus before, and we are only trying to fill in some gaps. If certain concepts in this problem set are unfamiliar to you, read your F.Sc. book and talk to the course staff.

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1. Let (x_0, y_0) be a point of the plane, and let L be the graph of the function $f(x) = mx+b$. Find the point $\tilde{x}$ such that the distance from (x_0, y_0) to $(\tilde{x}, f(\tilde{x}))$ is smallest. [Notice that minimizing this distance is the same as minimizing its square. This may simplify the computations somewhat.]

2. Use the inverse function theorem (in the new notation which emphasizes the local nature of the derivative) to find the derivative of $g(x) = \sin^{-1}(x)$.

3. Show that $f(x) = 2x + \cos x$ is one-to-one. What is the value of $f^{-1}(1)$?

4. Consider the function $f(x) = x + \lfloor x \rfloor$ on the domain $0 \leq x \leq 3$, where $\lfloor x \rfloor$ denotes the largest integer not greater than x. Either argue that $\int_0^3 f$ does not exist or compute the integral $\int_0^3 f$.

5. Compute the area of the region between the graphs of f and g over the interval $[a, b]$ where $f(x) = |x - 1|$, $g(x) = x^2 - 2x$, $a = 0$ and $b = 2$.

6. Use properties of the integral to compute the following in terms of π.

   (a) $\int_{-3}^{3} \sqrt{9 - x^2} dx$

   (b) $\int_0^2 \sqrt{1 - \frac{1}{4}x^2} dx$

   (c) $\int_{-2}^{2} (x - 3)\sqrt{4 - x^2} dx$

7. Carefully compute the following integrals or argue that the integral does not exist.

   (a) $\int_{-1}^{3} f$ where $f(x) = 1/x^2$

(b) $\int_{-1}^{1} f$ where $f(x) = 1/x^3$.

(c) $\int_{1}^{\infty} f$ where $f(x) = 1/x^2$.

(d) $\int_{\pi/4}^{3\pi/4} f$ where $f(x) = \sec(x)$.

8. If $f(1) = 12$, Df is continuous and $\int_1^4 Df = 14$, what is the value of $f(4)$.

9. Show that
$$0 \leq \int_5^{10} \frac{x^2}{x^4 + x^2 + 1} dx \leq 0.1$$
by comparing the integrand to a simpler function.

10. Using both forms of the change of variables theorem (by both versions we mean the one taught in F.Sc./A-level's and the one taught in this course), compute $\int_0^4 f$ where $f(y) = \sqrt{2y+1}$.

11. Bonus question! This question should help you understand the reason why is the definition in terms of upper and lower sums very convenient for mathematicians. We will use the "other" definition of integrability (the one used by engineers and scientists) to show that $f(x) = x$ is integrable over $[0, b]$ and its integral is
$$\int_0^b f = \frac{b^2}{2}$$
Definition of integrability in terms of Riemann sums:
$\int_0^b f = \frac{b^2}{2}$ means that given any number $\epsilon > 0$, there exists a corresponding number $\delta > 0$, such that for every partition P of $[0, b]$,
$$\|P\| < \delta \implies |\sum_{k=1}^{n} f(c_k)\Delta x_k - \frac{b^2}{2}| < \epsilon$$
where the partition $P : x_0 < x_1 < ... < x_{n-1} < x_n$, $x_0 = 0$ and $x_n = b$ with each $\Delta x_k < \delta$. Note that $\Delta x_k = x_k - x_{k-1}$ and $c_k \in [x_{k-1}, x_k]$.