Solution: Problem Set 4
Calculus 1

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Problem 1

Recall the statement of the fundamental theorem which guarantees the differentiability of $g(x) = \int_a^x f$ at x if f is *continuous* at x. If f is not continuous, the fundamental theorem does not tell us anything and we need to check the differentiability of g explicitly. Resorting to the definition of the derivative of g at $x = 0$, we have

$$\begin{aligned} g'(0) &= \lim_{\Delta x \to 0} \frac{g(\Delta x) - g(0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_0^{\Delta x} f(t)dt \end{aligned}$$

Now, if $\Delta x > 0$, then $f(t) = 2$ and

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_0^{\Delta x} f(t)dt &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_0^{\Delta x} 2dt \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}(2\Delta x) \\ &= \lim_{\Delta x \to 0} 2 \\ &= 2. \end{aligned}$$

If on the other hand $\Delta x < 0$ then $f(t) = t$ and

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_0^{\Delta x} f(t)dt &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_0^{\Delta x} tdt \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{(\Delta x)^2}{2} \\ &= \lim_{\Delta x \to 0} \frac{\Delta x}{2} \\ &= 0. \end{aligned}$$

We therefore conclude that the derivative of g at 0, which is a limit, does not exist. On the other hand, f is continuous at all x other than $x = 0$ and hence we can apply the fundamental theorem and say that

$$g'(x) = f(x) \quad \text{if} \quad x \neq 0.$$

However, remember that once we say that g is *differentiable* we mean that g is differentiable at every single point of the domain. Since this is not true, g is not a differentiable function on $\mathbb{R}$.

Problem 2

(a)

Since an anti-derivative of t^2 is $t^3/3$, by the corollary to the Fundamental theorem we have

$$\int_a^b t^2 dt = \frac{t^3}{3}\bigg|_a^b$$

and therefore,

$$f(x) = \int_1^{\cos(x)} t^2 dt = \frac{t^3}{3}\bigg|_1^{\cos(x)} = \frac{\cos^3(x)}{3} - \frac{\cos(1)}{3}.$$

(b)

From Part (a),

$$f(x) = \frac{\cos^3(x)}{3} - \frac{\cos(1)}{3}.$$

Applying the chain rule we get

$$\frac{df}{dx} = \frac{d}{dx}\left(\frac{\cos^3(x)}{3} - \frac{\cos(1)}{3}\right) = -\cos^2(x)\sin(x).$$

(c)

Since $g(x) = \int_1^x t^2 dt$, by letting $h(x) = \cos(x)$, we get

$$(g \circ h)(x) = g(h(x)) = g(\cos(x)) = \int_1^{\cos(x)} t^2 dt = f(x).$$

Differentiating both sides of the above equation, we get

$$D(g \circ h)(x) = Df(x).$$

Now applying the chain rule to the left-hand side of the equation above, we get

$$Df(x) = Dg(h(x))Dh(x). \tag{1}$$

We also know that

$$g(x) = \int_1^x t^2 dt.$$

Now applying the fundamental theorem we get

$$\frac{dg}{dx} = Dg(x) = x^2. \tag{2}$$

Since $h(x) = \cos(x)$, using (2) we get

$$Dg(h(x)) = [\cos(x)]^2 = \cos^2(x). \tag{3}$$

Also

$$Dh(x) = -\sin(x). \tag{4}$$

Now substituting (3) and (4) in (1) we get the same result as in Part (b), namely

$$Df(x) = \frac{df}{dx} = -\cos^2(x)\sin(x).$$

Problem 3

(a)

We wish to compute the function Df where the function f is defined as

$$f(x) = \int_0^{\alpha(x)} g(t)dt.$$

Thinking the same way as in the previous problem, let

$$h(x) = \int_0^{x} g(t)dt,$$

and by the fundamental theorem

$$Dh(x) = g(x).$$

Then

$$f(x) = (h \circ \alpha)(x) = h(\alpha(x)) = \int_0^{\alpha(x)} g(t)dt.$$

Now differentiating the above equation and applying the chain rule, we get

$$Df(x) = D(h \circ \alpha)(x) = Dh(\alpha(x))D\alpha(x),$$

and hence

$$Df(x) = g(\alpha(x))\alpha'(x).$$

The above equation tells us what the function does if the input variable is x. It is easy to see now that,

$$Df(y) = g(\alpha(y))\alpha'(y).$$

(b)

For a given x, let a be a real number, such that $\beta(x) \leq a \leq \alpha(x)$, then

$$\begin{aligned} f(x) &= \int_{\beta(x)}^{\alpha(x)} g(t)dt \\ &= \int_{\beta(x)}^{a} g(t)dt + \int_{a}^{\alpha(x)} g(t)dt \\ &= \int_{a}^{\alpha(x)} g(t)dt - \int_{a}^{\beta(x)} g(t)dt \end{aligned}$$

Now we differentiate the last equation at x and use the result obtained in the previous part to get

$$Df(x) = g(\alpha(x))\alpha'(x)) - g(\beta(x))\beta'(x).$$

And therefore, the function Df as a function of the variable z is given as

$$Df(z) = g(\alpha(z))\alpha'(z)) - g(\beta(z))\beta'(z).$$

Problem 4

Notice that

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

and $\frac{d}{dx}(\cos(x)) = -\sin(x)$, so if we let $u = \cos(x)$, then $du = -\sin(x)dx$. When $x = a, u = \cos(a)$ and when $x = b, u = \cos(b)$. Using these values we get

$$\begin{aligned}
\int_a^b \tan(x)dx &= \int_a^b \frac{\sin(x)}{\cos(x)}dx \\
&= -\int_{\cos(a)}^{\cos(b)} \frac{du}{u} \\
&= \ln(\cos(a)) - \ln(\cos(b)) \\
&= \ln\left[\frac{\cos(a)}{\cos(b)}\right]
\end{aligned}$$

This is probably the way you would have solved this question using your previous background. Let's do this question by the new method. Let

$$f(x) = \frac{1}{x}$$

and let

$$g(x) = \cos(x).$$

Then

$$g'(x) = -\sin(x).$$

We can clearly see that

$$(f \circ g)(x)g'(x) = \frac{1}{\cos(x)}(-\sin(x)) = -\tan(x).$$

From the theorem in the lecture notes

$$\int_a^b (f \circ g)(x)g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx.$$

Therefore

$$\begin{aligned}
\int_a^b -\tan(x)dx &= \int_{\cos(a)}^{\cos(b)} \frac{1}{x}dx \\
\int_a^b \tan(x)dx &= -\left[\ln(\cos(b)) - \ln(\cos(a))\right] \\
&= \ln\left[\frac{\cos(a)}{\cos(b)}\right]
\end{aligned}$$

Problem 5

(1)

Since the diagonal of the square runs from $-\sqrt{x}$ to $\sqrt{x}$, the length of the diagonal is $2\sqrt{x}$. We now want to determine the length of the side of the square. Let us call this length a. Then by the Pythagoras' theorem

$$a^2 + a^2 = (2\sqrt{x})^2.$$

Solving for a we get $a = \sqrt{2x}$. Therefore the area of the square at x is given by

$$A(x) = a^2 = 2x.$$

The volume is then given as

$$\begin{aligned} V &= \int_0^4 A(x)dx \\ &= \int_0^4 2x dx \\ &= \left[x^2\right]_0^4 \\ &= 16. \end{aligned}$$

(2)

The length of the diameter of the cross-sectional disk is

$$(2 - x^2) - x^2 = 2 - 2x^2.$$

Therefore the length of the radius of the cross sectional disk at x is given by

$$r(x) = 1 - x^2$$

and the area of the cross-sectional disk at x is therefore

$$A(x) = \pi r(x)^2 = \pi(1 - x^2)^2.$$

The volume can now be calculated as

$$\begin{aligned} V &= \int_{-1}^1 A(x)dx \\ &= \pi \int_{-1}^1 (1 - x^2)^2 dx \\ &= \pi \int_{-1}^1 (1 - 2x^2 + x^4) dx \\ &= \frac{16}{15}\pi \end{aligned}$$

(5)

(a)

The area of an equilateral triangle with side of length a is given by

$$\Delta = \frac{\sqrt{3}a^2}{4}.$$

The derivation of the above formula is left as an exercise. For this question

$$a = 2\sqrt{\sin x}$$

and hence the area of the cross-sectional triangle at x is given by

$$A(x) = \frac{\sqrt{3}a^2}{4} = \frac{4\sqrt{3}\sin x}{4} = \sqrt{3}\sin x.$$

The volume is therefore given as

$$\begin{aligned} V &= \int_0^\pi A(x)dx \\ &= \sqrt{3}\int_0^\pi \sin x dx \\ &= 2\sqrt{3} \end{aligned}$$

(b)

The area of the cross-sectional square is given by

$$A(x) = (2\sqrt{\sin x})(2\sqrt{\sin x}) = 4\sin x.$$

The volume is therefore given as

$$\begin{aligned} V &= \int_0^\pi A(x)dx \\ &= 4\int_0^\pi \sin x dx \\ &= 8 \end{aligned}$$

(10)

The length of one leg of the triangle at y is given as $2\sqrt{1-y^2}$. The area of the cross-sectional triangle, which is an isosceles triangle, is given as

$$A(y) = \frac{(2\sqrt{1-y^2})(2\sqrt{1-y^2})}{2} = 2(1-y^2)$$

The volume is hence given as

$$\begin{aligned} V &= \int_{-1}^1 A(y)dy \\ &= 2\int_{-1}^1 (1-y^2)dy \\ &= \frac{8}{3} \end{aligned}$$

(12)

If we rotate the pyramid anti-clockwise, we can see that the cross sections of this tilted pyramid are squares of sides varying linearly from 0 to 3 as x varies from 0 to 5. Therefore the side of the cross-sectional square, as a function of x, is given as

$$l(x) = \frac{3}{5}x.$$

The area would be

$$A(x) = l^2(x) = \frac{9}{25}x^2.$$

The volume is then given as

$$\begin{aligned} V &= \int_0^5 A(x)dx \\ &= \frac{9}{25}\int_0^5 x^2 dx \\ &= 15 \end{aligned}$$